RICE'S THEOREM

Rice's Theorem

- Def:
  - A property of the Turing-recognizable languages is simply a subset of all Turing-recognizable languages.

- A property is trivial if it is either empty or is all Turing-recognizable languages. Otherwise, it is nontrivial.

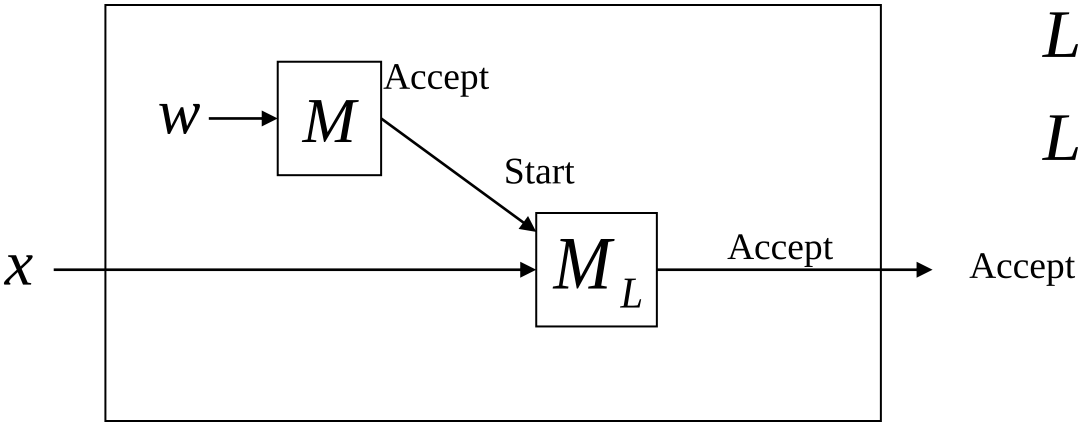
- Note that the empty property $\emptyset$ is different from the property $\{\emptyset\}$.

- Thm:
  - Every nontrivial property of the Turing-recognizable languages is undecidable.

- Pf:
  Let C be a nontrivial property.
  Assume $\varnothing$, otherwise we can consider .
  Since C is nontrivial, there exists a non-empty
  language $L \in C$. $\notin C$

  $\overline{C}$

- Let M_L be a TM accepting (or recognizing) L. Given $\langle M, w \rangle$, construct the following TM M':



$$L(M') = \varnothing \text{ if } \langle M, w \rangle \notin A_{TM},$$
$$L(M') = L \text{ if } \langle M, w \rangle \in A_{TM}$$

- Thus, if we could decide $L(M') \in C$ or not ,then we could decide A_{TM}.
- Therefore, C is undecidable.

- Eg. The following are undecidable:
  - 1.
  - 2.
  - 3.
  - 4.

$$E_{TM} = \{\langle M \rangle \mid L(M) = \varnothing\}.$$

$$FINITE_{TM} = \{\langle M \rangle \mid L(M) \text{ is finite}\}.$$

$$REGULAR_{TM} = \{\langle M \rangle \mid L(M) \text{ is regular}\}.$$

$$CFL_{TM} = \{\langle M \rangle \mid L(M) \text{ is a context-free language}\}.$$

- Eg. How about the following:
  - 1.
  - 2.

$$FIVE = \{\langle M \rangle \mid M \text{ has five states}\}.$$

$$MOVE_5 = \{\langle M \rangle \mid M \text{ makes at least 5 moves}\}.$$